Maximal Functions and Almost-Everywhere Convergence

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Central Question

Does a given sequence of functions have a limit almost-everywhere (with respect to an underlying measure), and if so, what is that limit?

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We take sequences of the form $T_n f$ where $T_n : X \to \mathcal{M}(\Omega)$, X is a Banach space and $\mathcal{M}(\Omega)$ are $\mathbb{R}$ or $\mathbb{C}$ valued measurable functions defined on a measure space Ω.

$$T_\infty f := \lim_{n\to\infty} T_n f \ (?)$$

Maximal Functions

Definition (Maximal Functions)

For each $f \in X$ and $x \in \Omega$, we define

$$T^* f(x) := \sup_n |T_n f(x)| \ .$$

If $T^* f$ is measurable, we call it a *maximal function* (on Ω). T^* is called a *maximal operator* (from X to Ω).

The Magic of Maximal Functions

Theorem (The Magic of Maximal Functions)
Let T_n be a sequence of subadditive operators from X to Ω and T^* be the associated maximal operator. Suppose that:

(H1) *There exists a dense subset $X_0 \subseteq X$ where for each $f \in X_0$, $T_n f$ has a measurable limit for a.e. $x \in \Omega$.*

(H2) *T^* satisfies the C–δ Hypothesis*

Then, we have:

(i) *For every $f \in X$, $T_n f(x)$ has a measurable limit, $T_\infty f(x)$, for a.e. $x \in \Omega$. T_∞ is an operator from X to Ω.* **(Existence)**

(ii) *The operator T_∞ is continuous in measure: that is, if $f_n \to f$ in X, then $T_\infty f_n(x) \to T_\infty f(x)$ in measure.* **(Identification)**

The C–δ Hypothesis

Definition (The C–δ Hypothesis)
Let T^* be a maximal operator from X to Ω. Define for $C, \delta > 0$,

$$M(\delta, C) := \sup_{f;\|f\|\leq\delta} \mu(T^* f(x) > C) \ .$$

We say that T^* satisfies *the C–δ hypothesis* if, for all $C > 0$, we have that

$$\lim_{\delta\to 0^+} M(\delta, C) = 0 \ .$$

Maximal Inequalities

Definition (Maximal Inequalities)
Let T^* be a maximal operator from X to Ω. A *maximal inequality* is a bound of the form

$$\mu(T^* f > C) \leq Q\left(\frac{\|f\|_X}{C}\right)$$

where $\lim_{x \to 0^+} Q(x) = 0$.

Often, $Q(x) \simeq x^p$. When Q is of this form, T^* satisfying a maximal inequality is equivalent to T^* being a bounded operator into weak-L^p.

The C–δ Hypothesis is HARD

The Magic of Maximal Functions requires three things:

- Subadditive operators T_n
- (H1) $\lim_n T_n f$ exists for $f \in X_0$, $\overline{X_0} = X$
- (H2) T^* satisfies a Maximal Inequality

Once these three hypotheses are satisfied, the proof is done.

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- (H2) T^* satisfies a Maximal Inequality (ALWAYS Hard)

Once these three hypotheses are satisfied, the proof is done.

Example: Carleson's Theorem

Theorem (Carleson–Hunt)

Let $f \in L^p(\mathbb{T})$, $1 < p < \infty$. *Then,*

$$\sum_{-n}^{n} \widehat{f}_k e^{2\pi ikx} =: S_n[f] \to f \ , \ \textit{Lebesgue a.e.}$$

($\widehat{f}_k$ denotes the kth Fourier coefficient of f)

S_n are additive, $\mathcal{C}^2$ functions are dense in L^p.

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Theorem (L. Carleson, R. Hunt, 1967)
We define the Carleson maximal operator by

$$M_{\mathrm{C}} f := \sup_{n \in \mathbb{N}} |S_n[f]| \ .$$

M_{C} *is a bounded operator from* L^p *to* L^p_w *for* $1 < p < \infty$.

Example 2: Lebesgue Differentiation Theorem

Theorem (Generalized Lebesgue Differentiation Theorem)

Let μ be a σ-finite Borel measure on $\mathbb{R}$, and let $f \in L^1(\mathbb{R}, d\mu)$. Then we have

$$\lim_{\epsilon \to 0^+} \frac{1}{\mu(B_\epsilon(x))} \int_{B_\epsilon(x)} f(y) d\mu(y) = f(x)$$

for μ.a.e. $x \in \mathbb{R}$.

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$$T_\epsilon f := \frac{1}{\mu(B_\epsilon(x))} \int_{B_\epsilon(x)} f(y) d\mu(y)$$

T_ϵ are additive, uniformly continuous functions are dense in L^p.

Example 2: Lebesgue Differentiation Theorem

Definition (Generalized Hardy–Littlewood Maximal Function)

Let μ be a σ-finite Borel measure on $\mathbb{R}$. For each $f \in L^1(\mathbb{R}, d\mu)$ define

$$M_{\mathrm{HL}}^{\mu} f := \sup_{\epsilon > 0} \frac{1}{\mu(B_\epsilon(x))} \int_{B_\epsilon(x)} |f(y)| d\mu(y) \ .$$

Theorem (Generalized Hardy-Littlewood Maximal Inequality)

For each $C > 0, f \in L^1(\mathbb{R}, d\mu)$,

$$\mu(M_{HL}^{\mu} f > C) \leq \frac{2 \|f\|_1}{C} \ .$$

(Based on Croft-Garsia, as opposed to Vitali for Lebesgue measure)

Example 3: The Strong Law of Large Numbers

Theorem (The Strong Law of Large Numbers)

Let $(\Omega, d\mu)$ be a probability space, and let f_n be iidrv's with the additional condition that $\mathbb{E}(f_n) < \infty$ for each n. Then,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f_j(\omega) = \mathbb{E}(f_1) \quad \mu\text{-almost surely.}$$

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Let $f = (f_1, f_2, \dots)$ and define

$$Av_n[f] := \frac{1}{n} \sum_{1}^{n} f_j$$

Av_n are linear, $\lim_{n\to\infty} Av_n f = \mathbb{E}(f_1)$ for $f \in I \oplus I^{\perp}$ where I is the set of bounded, Bernoulli-shift-invariant functions.

Example 3: The Strong Law of Large Numbers

Define

$$M_{\mathrm{E}}f := \sup_n |Av_n[f]|$$

Theorem

Let $(\Omega, d\mu)$ *be a probability space and* $f_k \in L^1(\Omega)$ *(that is,* $\mathbb{E}(f_k) < \infty$*). For any* $\alpha > 0$ *, we have that*

$$\mu(M_{\mathrm{E}}f > \alpha) \leq \frac{\mathbb{E}(f_1)}{\alpha}.$$

(Based on the Hopf–Kakutani–Yoshida Maximal Ergodic Theorem)